

## Correlated random effects: iid1d, iid2d, iid3d, iid4d and iid5d

This model is available for dimensions  $p = 1, 2, 3, 4$  and  $5$ . We describe in detail the case for  $p = 2$ , and then the changes required for  $p = 1, p = 3, p = 4$  and  $p = 5$

### Parametrization

The 2-dimensional Normal-Wishard model is used if one want to define two vectors of “random effects”,  $u$  and  $v$ , say, for which  $(u_i, v_i)$  are iid bivariate Normals

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{W}^{-1})$$

where the covariance matrix  $\mathbf{W}^{-1}$  is

$$\mathbf{W}^{-1} = \begin{pmatrix} 1/\tau_a & \rho/\sqrt{\tau_a\tau_b} \\ \rho/\sqrt{\tau_a\tau_b} & 1/\tau_b \end{pmatrix} \quad (1)$$

and  $\tau_a, \tau_b$  and  $\rho$  are the hyperparameters.

Note that  $\rho$  is the correlation coefficient, and that  $\tau_a$  and  $\tau_b$  are the *marginal precisions*, *not* the elements in the precision matrix.

For these models the precision matrix  $\mathbf{W}$  is Wishart distributed

$$\mathbf{W} \sim \text{Wishart}_p(r, \mathbf{R}^{-1}), \quad p = 2$$

with density

$$\pi(\mathbf{W}) = c^{-1} |\mathbf{W}|^{(r-(p+1))/2} \exp \left\{ -\frac{1}{2} \text{Trace}(\mathbf{W}\mathbf{R}) \right\}, \quad r > p + 1$$

and

$$c = 2^{(rp)/2} |\mathbf{R}|^{-r/2} \pi^{(p(p-1))/4} \prod_{j=1}^p \Gamma((r+1-j)/2).$$

Then,

$$\text{E}(\mathbf{W}) = r\mathbf{R}^{-1}, \quad \text{and} \quad \text{E}(\mathbf{W}^{-1}) = \mathbf{R}/(r - (p + 1)).$$

### Hyperparameters

The hyperparameters are

$$\theta = (\log \tau_a, \log \tau_b, \tilde{\rho})$$

where

$$\rho = 2 \frac{\exp(\tilde{\rho})}{\exp(\tilde{\rho}) + 1} - 1$$

The prior-parameters are

$$(r, R_{11}, R_{22}, R_{12})$$

where

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

and  $r_{12} = R_{21}$  due to symmetry.

The `inla` function reports posterior distribution for the hyperparameters  $\tau_a, \tau_b, \rho$  in equation (1).

The prior for  $\theta$  is **fixed** to be `wishart2d`

## Specification

The model `iid2d` is specified as

$$y \sim f(i, \text{model}="iid2d", n = \langle \text{length} \rangle) + \dots$$

and the `iid2d` model is represented internally as one vector of length  $n$ ,

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m)$$

where  $n = 2m$ , and  $n$  is the (required) argument in `f()`.

For this model the argument `constr=TRUE` is interpreted as

$$\sum u_i = 0, \quad \text{and} \quad \sum v_i = 0.$$

## Hyperparameter spesification and default values

`doc` Gaussian random effect in dim=2 with Wishart prior

`hyper`

`theta1`

```
hyperid 26001
name log precision1
short.name prec1
initial 4
fixed FALSE
prior wishart2d
param 4 1 1 0
to.theta function(x) log(x)
from.theta function(x) exp(x)
```

`theta2`

```
hyperid 26002
name log precision2
short.name prec2
initial 4
fixed FALSE
prior none
param
to.theta function(x) log(x)
from.theta function(x) exp(x)
```

`theta3`

```
hyperid 26003
name logit correlation
short.name cor
initial 4
fixed FALSE
```

```

    prior none
    param
    to.theta function(x) log((1 + x) / (1 - x))
    from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1

constr FALSE

nrow.ncol FALSE

augmented TRUE

aug.factor 1

aug.constr 1 2

n.div.by 2

n.required TRUE

set.default.values TRUE

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```

## Example

In these examples we demonstrate the use of the iid2d-model, with observations that are without noise (essentially).

```

n = 1000
N = 2*n

## need it to simulate data
library(mvtnorm)

if (TRUE)
{
  ## first example - each variable in the correlated pair has its own row in data

  #Using fixed covariance matrix
  rho = 0.5
  ## set variances
  Sigma = matrix(c(1/1, NA, NA, 1/2), 2, 2)
  ## and the correlation
  Sigma[1,2] = Sigma[2,1] = rho*sqrt(Sigma[1,1]*Sigma[2,2])

  y = yy = rmvnorm(n, sigma=Sigma)
  y = c(y[,1], y[,2])

  i = 1:N
  formula = y ~ f(i, model="iid2d", n=N)

  r = inla(formula, data = data.frame(i,y),
           control.family=list(initial=10,fixed=TRUE))

```

```

print(summary(r))

print(1/diag(cov(yy)))
print(cor(yy)[1,2])
}

if (TRUE)
{
  ## second example - both correlated variables occur in the same row of data

  #drawing covariance matrix from hyperprior
  #Let's specify non-default values, expecting strong covariance
  r = 4
  R11 = 5
  R22 = 4
  R12 = 3

  R = matrix(c(R11,R12,R12,R22), 2, 2)
  #Take a single sample from wishart_2(r,R^-1)
  W = rWishart(1,r,solve(R))[,1]
  Sigma = solve(W) #Compute the covariance matrix

  y = yy = rmvnorm(n, sigma=Sigma)
  z = rnorm(n)
  zz = rnorm(n)
  y = y[,1] + z*y[,2] + zz
  i = 1:n
  j = n + 1:n
  formula = y ~ f(i, model="iid2d", n=N) + f(j,z,copy="i") + zz

  r = inla(formula, data = data.frame(i,j,y,z,zz),
           control.family=list(initial=10,fixed=TRUE),keep=T)
  print(summary(r))

  #The params as in the Sigma matrix
  print(1/diag(Sigma))
  print(cov2cor(Sigma)[1,2])

  #The params as seen in data
  print(1/diag(cov(yy)))
  print(cor(yy)[1,2])
}

```

## The case $p = 1$

For  $p = 1$  the hyperparameter is the marginal precision

$$\theta = \log \tau_1$$

The prior is fixed to be `wishart1d` with parameters

$$param = r \ R_{11}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} \end{bmatrix}$$

## Hyperparameter spesification and default values

doc Gaussian random effect in dim=1 with Wishart prior

hyper

theta

hyperid 25001

name precision

short.name prec

initial 4

fixed FALSE

prior wishart1d

param 2 1e-04

to.theta function(x) log(x)

from.theta function(x) exp(x)

constr FALSE

nrow.ncol FALSE

augmented FALSE

aug.factor 1

aug.constr

n.div.by

n.required FALSE

set.default.values TRUE

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## The case $p = 3$

For  $p = 3$  the hyperparameters are

$$\theta = (\log \tau_1, \log \tau_2, \log \tau_3, \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{23})$$

The prior is fixed to be `wishart3d` with parameters

$$param = r \ R_{11} \ R_{22} \ R_{33} \ R_{12} \ R_{13} \ R_{23}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ R_{13} & R_{23} & R_{33} \end{bmatrix}$$

The reported hyperparameters are the marginal precisions  $\tau_1$ ,  $\tau_2$  and  $\tau_3$  and the correlations  $\rho_{12}$ ,  $\rho_{13}$  and  $\rho_{23}$ .

In this case, the internal representation is given as

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m)$$

where  $n = 3m$  is a required argument, and where  $(u_i, v_i, w_i)$  are trivariate iid Normal.

## Hyperparameter spesification and default values

doc Gaussian random effect in dim=3 with Wishart prior

hyper

theta1

```
hyperid 27001
name log precision1
short.name prec1
initial 4
fixed FALSE
prior wishart3d
param 7 1 1 1 0 0 0
to.theta function(x) log(x)
from.theta function(x) exp(x)
```

theta2

```
hyperid 27002
name log precision2
short.name prec2
initial 4
fixed FALSE
prior none
param
to.theta function(x) log(x)
from.theta function(x) exp(x)
```

theta3

```
hyperid 27003
name log precision3
short.name prec3
initial 4
fixed FALSE
prior none
param
```

```

    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta4
  hyperid 27004
  name logit correlation12
  short.name cor12
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta5
  hyperid 27005
  name logit correlation13
  short.name cor13
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta6
  hyperid 27006
  name logit correlation23
  short.name cor23
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
constr FALSE
nrow.ncol FALSE
augmented TRUE
aug.factor 1
aug.constr 1 2 3
n.div.by 3
n.required TRUE
set.default.values TRUE
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```

## The case $p = 4$

For  $p = 4$  the hyperparameters are

$$\theta = (\log \tau_1, \log \tau_2, \log \tau_3, \log \tau_4, \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{14}, \tilde{\rho}_{23}, \tilde{\rho}_{24}, \tilde{\rho}_{34})$$

The prior is fixed to be **wishart4d** with parameters

$$param = r \ R_{11} \ R_{22} \ R_{33} \ R_{44} \ R_{12} \ R_{13} \ R_{14} \ R_{23} \ R_{24} \ R_{34}$$

where

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{12} & R_{22} & R_{23} & R_{24} \\ R_{13} & R_{23} & R_{33} & R_{34} \\ R_{14} & R_{24} & R_{34} & R_{44} \end{bmatrix}$$

The reported hyperparameters are the marginal precisions  $\tau_1, \tau_2, \tau_3$  and  $\tau_4$ , and the correlations  $\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}$  and  $\rho_{34}$ .

In this case, the internal representation is given as

$$(u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m, x_1, x_2, \dots, x_m)$$

where  $n = 4m$  is a required argument, and where  $(u_i, v_i, w_i, x_i)$  are fourvariate iid Normal.

## Hyperparameter spesification and default values

**doc** Gaussian random effect in dim=4 with Wishart prior

**hyper**

**theta1**

```
hyperid 28001
name log precision1
short.name prec1
initial 4
fixed FALSE
prior wishart4d
param 11 1 1 1 1 0 0 0 0 0 0
to.theta function(x) log(x)
from.theta function(x) exp(x)
```

**theta2**

```
hyperid 28002
name log precision2
short.name prec2
initial 4
fixed FALSE
prior none
param
to.theta function(x) log(x)
```



```

    from.theta function(x) exp(x)
theta3
    hyperid 28003
    name log precision3
    short.name prec3
    initial 4
    fixed FALSE
    prior none
    param
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta4
    hyperid 28004
    name log precision4
    short.name prec4
    initial 4
    fixed FALSE
    prior none
    param
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta5
    hyperid 28005
    name logit correlation12
    short.name cor12
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1 + x) / (1 - x))
    from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta6
    hyperid 28006
    name logit correlation13
    short.name cor13
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1 + x) / (1 - x))
    from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta7
    hyperid 28007
    name logit correlation14

```

```

short.name cor14
initial 0
fixed FALSE
prior none
param
to.theta function(x) log((1 + x) / (1 - x))
from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta8
  hyperid 28008
  name logit correlation23
  short.name cor23
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta9
  hyperid 28009
  name logit correlation24
  short.name cor24
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta10
  hyperid 28010
  name logit correlation34
  short.name cor34
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
constr FALSE
nrow.ncol FALSE
augmented TRUE
aug.factor 1
aug.constr 1 2 3 4

```

```

n.div.by 4
n.required TRUE
set.default.values TRUE
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```

The case  $p = 5$

The case  $p = 5$  follows by a direct extension of  $p = 3$  and  $p = 4$ , and is therefore not included.

### Hyperparameter specification and default values

doc Gaussian random effect in dim=5 with Wishart prior

hyper

```

theta1
  hyperid 29001
  name log precision1
  short.name prec1
  initial 4
  fixed FALSE
  prior wishart5d
  param 16 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta2
  hyperid 29002
  name log precision2
  short.name prec2
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta3
  hyperid 29003
  name log precision3
  short.name prec3
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)

```

```

    from.theta function(x) exp(x)
theta4
  hyperid 29004
  name log precision4
  short.name prec4
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta5
  hyperid 29005
  name log precision5
  short.name prec5
  initial 4
  fixed FALSE
  prior none
  param
  to.theta function(x) log(x)
  from.theta function(x) exp(x)
theta6
  hyperid 29006
  name logit correlation12
  short.name cor12
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta7
  hyperid 29007
  name logit correlation13
  short.name cor13
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta8
  hyperid 29008
  name logit correlation14

```

```

short.name cor14
initial 0
fixed FALSE
prior none
param
to.theta function(x) log((1 + x) / (1 - x))
from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta9
  hyperid 29009
  name logit correlation15
  short.name cor15
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta10
  hyperid 29010
  name logit correlation23
  short.name cor23
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta11
  hyperid 29011
  name logit correlation24
  short.name cor24
  initial 0
  fixed FALSE
  prior none
  param
  to.theta function(x) log((1 + x) / (1 - x))
  from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta12
  hyperid 29012
  name logit correlation25
  short.name cor25
  initial 0
  fixed FALSE
  prior none

```

```

    param
    to.theta function(x) log((1 + x) / (1 - x))
    from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta13
    hyperid 29013
    name logit correlation34
    short.name cor34
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1 + x) / (1 - x))
    from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta14
    hyperid 29014
    name logit correlation35
    short.name cor35
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1 + x) / (1 - x))
    from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1
theta15
    hyperid 29015
    name logit correlation45
    short.name cor45
    initial 0
    fixed FALSE
    prior none
    param
    to.theta function(x) log((1 + x) / (1 - x))
    from.theta function(x) 2 * exp(x) / (1 + exp(x)) - 1

constr FALSE

nrow.ncol FALSE

augmented TRUE

aug.factor 1

aug.constr 1 2 3 4 5

n.div.by 5

n.required TRUE

set.default.values TRUE

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```

## Notes

The model `iid1d` is similar to the model `iid` (and included for completeness only). The prior for `iid1d` is fixed to be Wishart-distributed, which reduces to a Gamma-distribution for the precision with parameters

$$a = r/2 \quad \text{and} \quad b = R_{11}/2$$

hence

```
y ~ f(i, model="iid1d", hyper = list(theta=list(param=c(3, 4))))
```

is equivalent to

```
y ~ f(i, model="iid", hyper = list(theta=list(param=c(1.5, 2), prior="loggamma")))
```