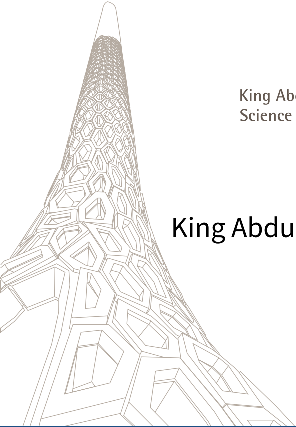
The Role of GMRFs in Statistical Modeling and Scalable Bayesian Inference (part I)



جامعة الملك عبدالله
للعلوم والتقنية

King Abdullah University of
Science and Technology

Håvard Rue
King Abdullah University of Science and Technology
Saudi Arabia

May 2026

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- I had to email Miguel, saying
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- part I and part II.
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- Correlation
- GMRFs
- Why GMRFs?
- Why 'Markov'?
- When is "GMRF + GMRF = GMRF"?
- What about 'non-Markov'?

Russian mathematician Andrey Markov

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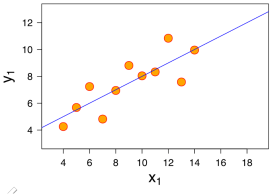
Correlation: definition

Tightly connected to the multivariate Normal distribution

- Need the concept of *correlation*
- Two random variables U and V, with zero mean and unit variance:

$$\mathrm{Corr}(U, V) \stackrel{\text{def}}{=} \mathrm{E}(UV)$$

- Was "invented" in regression late 1800
- Often motivated using regression
- Describes "linear dependence" only



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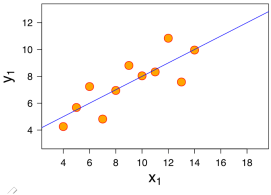

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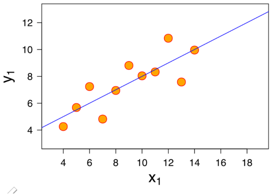

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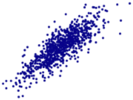

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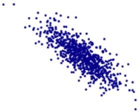

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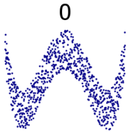

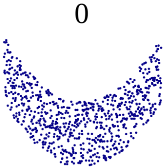

Multivariate Normal distribution

- Random vector $(x_1, \ldots, x_n)$
- (Scaled and centred) density

$$\pi(\boldsymbol{x}) \propto |\boldsymbol{C}|^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{C}^{-1}\boldsymbol{x}\right)$$

- $\boldsymbol{C}$ is the (SPD) correlation matrix
- $\text{diag}(\boldsymbol{C}) = \boldsymbol{I}$
- $C_{ij} = \text{E}(x_i x_j)$
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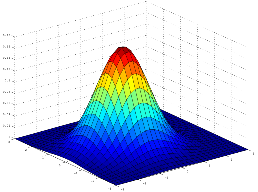

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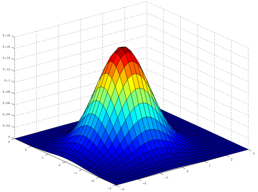

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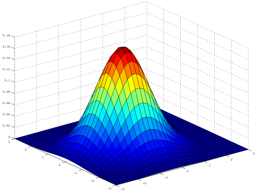

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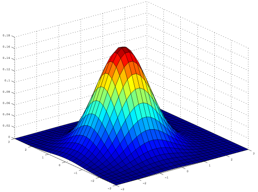

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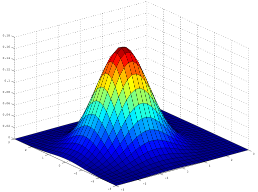

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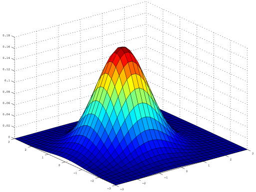


How to think about correlation?

- dim$=2$ with correlation ρ, then

$$E(x_1|x_2) = \rho x_2 \qquad \mathrm{Var}(x_1|x_2) = 1-\rho^2$$

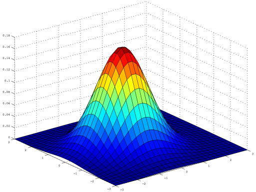
- What value of ρ is “half-correlation”?
- The natural “distance” is $\sqrt{\mathrm{Var}(x_1|x_2)}$, hence

$$\sqrt{1-\rho^2} = \frac{1}{2}$$

gives $\rho = \sqrt{3}/2 = 0.866\ldots$

- A “better” parameterisation:

$$\widetilde{\rho} = 1 - \mathrm{sign}(\rho)\sqrt{1-\rho^2}$$



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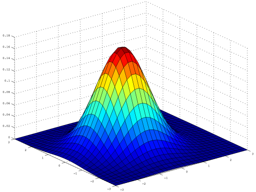

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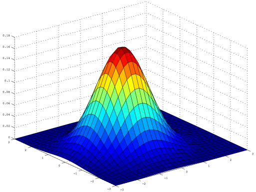

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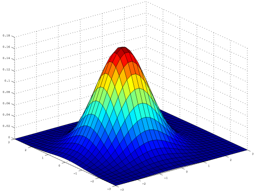


Conditional independence

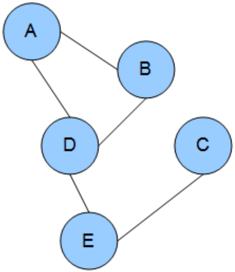
- Conditional independence of x_i and x_j, equals independence in

$$\pi(x_i, x_j \mid \boldsymbol{x}_{-ij})$$

- The result is, with $\boldsymbol{Q} = \boldsymbol{C}^{-1}$, that

$$Q_{ij} = 0 \Leftrightarrow x_i \perp x_j \mid \boldsymbol{x}_{-ij}$$

- This is potentially very very useful, as
  - If $\boldsymbol{Q}$ is fully connected then $\boldsymbol{C}$ is dense (Cayley–Hamilton thm)
  - Describe dependence by a sparse $\boldsymbol{Q}$
  - Sparse matrices: faster computations



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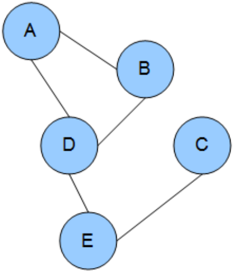

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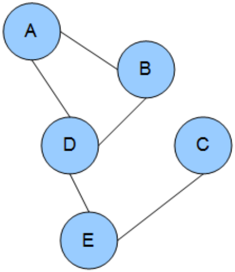

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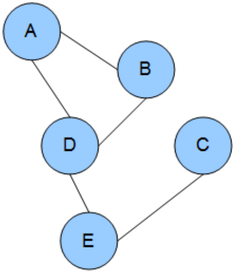

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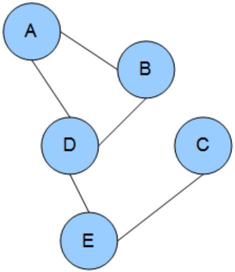

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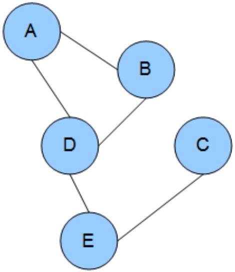


Example: Markov processes in time

- Auto-regressive process of order 1

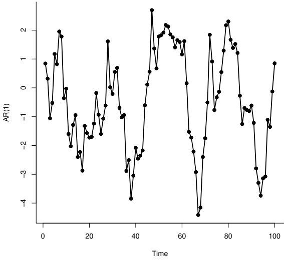
$$x_t = \phi x_{t-1} + \epsilon_t, \quad 0 \leq \phi < 1, \quad t = 1, \ldots, n$$

  with Gaussian noise ϵ_t

- Linear graph
- $\mathbf{Q}$ is tridiagonal (factorisation cost $\mathcal{O}(n)$)
- $\mathbf{C}$ is $\phi^{-|i-j|}$ (exp-correlation function)
- Cont.time: Ornstein-Uhlenbeck process

$$dx_t = -\theta\, x_t\, dt + dW_t,$$

  Wiener process W_t



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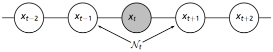

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$$\begin{pmatrix} 1 & \phi & \phi^2 & \phi^3 & \phi^4 & \phi^5 & \phi^6 \\ \phi & 1 & \phi & \phi^2 & \phi^3 & \phi^4 & \phi^5 \\ \phi^2 & \phi & 1 & \phi & \phi^2 & \phi^3 & \phi^4 \\ \phi^3 & \phi^2 & \phi & 1 & \phi & \phi^2 & \phi^3 \\ \phi^4 & \phi^3 & \phi^2 & \phi & 1 & \phi & \phi^2 \\ \phi^5 & \phi^4 & \phi^3 & \phi^2 & \phi & 1 & \phi \\ \phi^6 & \phi^5 & \phi^4 & \phi^3 & \phi^2 & \phi & 1 \end{pmatrix}$$

Example: Markov processes in time

- Auto-regressive process of order 1

$$x_t = \phi x_{t-1} + \epsilon_t, \quad 0 \leq \phi < 1, \quad t = 1, \ldots, n$$

with Gaussian noise ϵ_t

- Linear graph
- $\boldsymbol{Q}$ is tridiagonal (factorisation cost $\mathcal{O}(n)$)
- $\boldsymbol{C}$ is $\phi^{-|i-j|}$ (exp-correlation function)
- Cont.time: Ornstein-Uhlenbeck process

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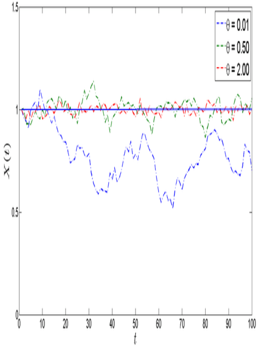

How does correlation work?

- Let

$$x_1 = z_1 + \mu$$

$$x_2 = z_2 + \mu$$

- z_1, z_2, μ are independent zero mean Gaussian's
- $\text{Var}(\mu) = \sigma^2$, $\text{Var}(z_i) = 1$
- Then

$$\text{Corr}(x_1, x_2) = \frac{\sigma^2}{\sigma^2 + 1}$$

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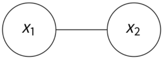

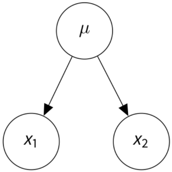

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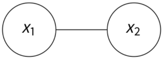

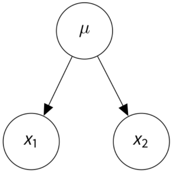

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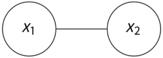

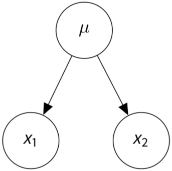

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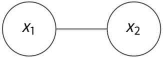

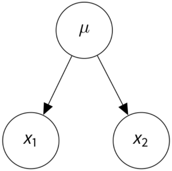

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- A Gaussian Markov Random Field is a multivariate Gaussian

$$\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{-1})$$

  where $\mathbf{Q}$ is the precision matrix ($= \mathbf{\Sigma}^{-1}$)
- $x_i \perp x_j | \mathbf{x}_{-ij} \iff Q_{ij} = 0$
- Form a graph $\mathcal{G}$ with an edge between i and j if $Q_{ij} \neq 0$.
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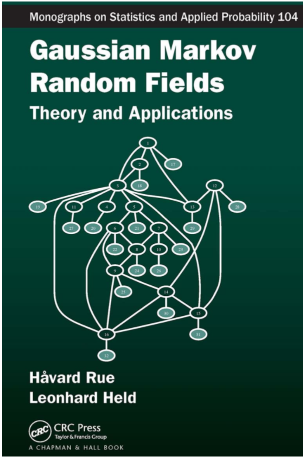

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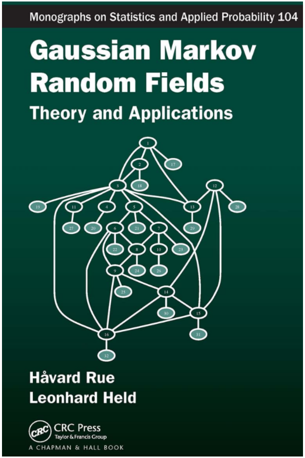

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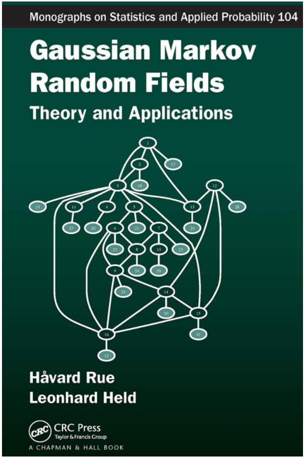

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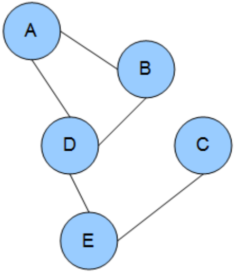

What are good "operations" for GMRFs?

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12}^T & \boldsymbol{\Sigma}_{22} \end{pmatrix} \qquad \boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \boldsymbol{Q}_{12}^T & \boldsymbol{Q}_{22} \end{pmatrix}$$

Conditioning $\boldsymbol{x}_1 | \boldsymbol{x}_2$

$$\text{Var}(\boldsymbol{x}_1 | \boldsymbol{x}_2) = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^T \qquad \text{Prec}(\boldsymbol{x}_1 | \boldsymbol{x}_2) = \boldsymbol{Q}_{11}$$

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then we observe

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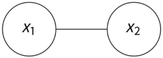

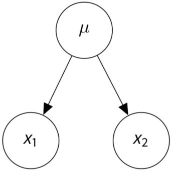

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- More focus on a “generative” model, where
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Example: dimension $= 1$

Time-series literature is, at best, notoriously confusing

- Auto-regressive processes

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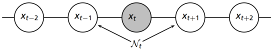

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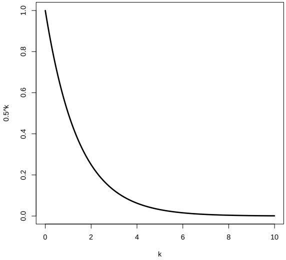
$$\text{Corr}(x_t, x_{t+s}) = \phi^{|s|}$$



- Another viewpoint

$$\text{E}(x_t | \boldsymbol{x}_{-t}) = \frac{\phi}{1+\phi^2}(x_{t-1} + x_{t+1})$$

with some conditional variance

Example: dimension$=1$

Time-series literature is, at best, notoriously confusing

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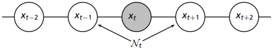

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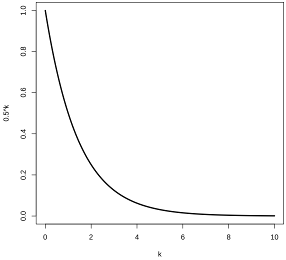

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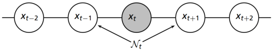

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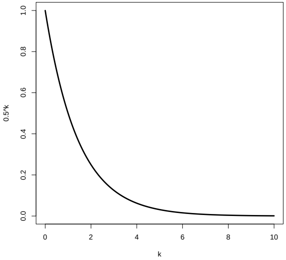

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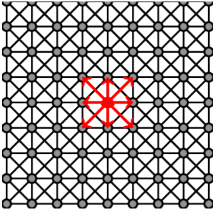
Example: dimesion$=2$

"standard" model in spatial statistics++ with regional data



- Traditionally written as

$$\pi(\boldsymbol{x}) \propto \tau^{\frac{n-k}{2}} \exp(-\frac{\tau}{2}\sum_{i\sim j} w_{ij}(x_i - x_j)^2)$$

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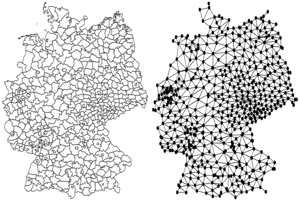

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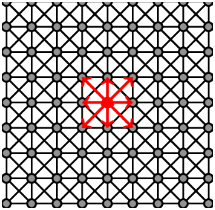

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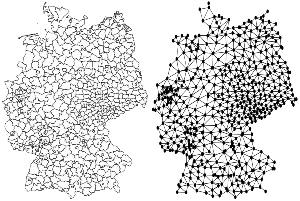


Additive regression models/GLM++

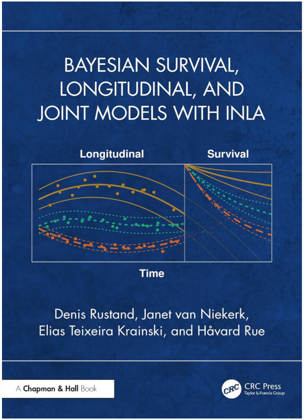
- Linear predictor

$$\eta_i = \mu + \sum_j \beta_j z_{ij} + \boldsymbol{f}_{1i} + \boldsymbol{f}_{2i} + \cdots$$

with observations

$$y_i | \ldots \sim \pi(y_i | \eta_i, \ldots)$$

- $\boldsymbol{f}_k$ is a Gaussian process, think times-series, spatial-model, spline, some kind of dependent 'random effect', measurement error model, etc...



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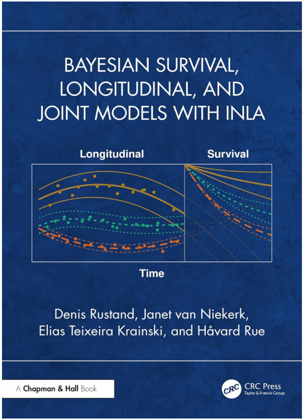

Key observation

- If each $\boldsymbol{f}_k$ is a GMRF, then

$$\boldsymbol{x} = (\beta_1, \beta_2, \ldots, \boldsymbol{f}_1, \boldsymbol{f}_2, \ldots)$$

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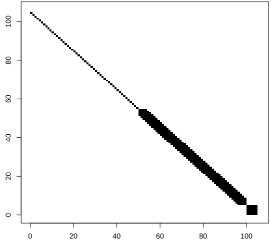

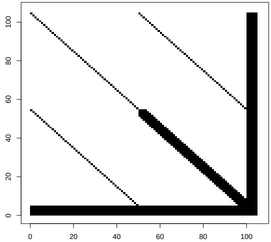

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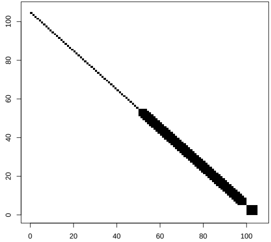

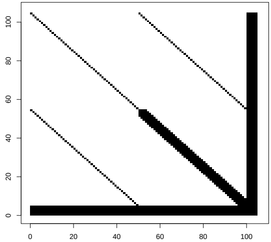

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GMRFs is closed

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- conditioning

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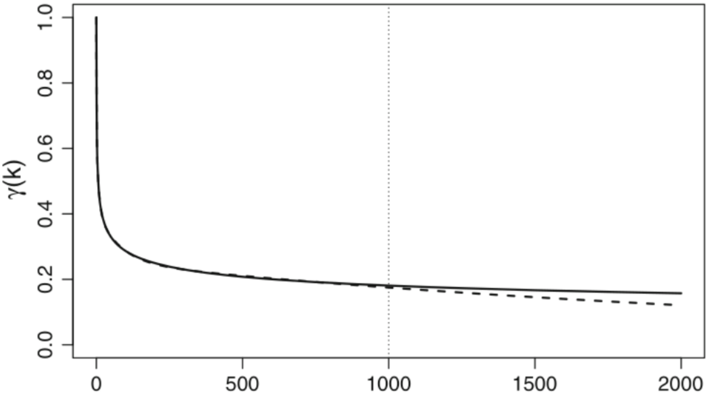

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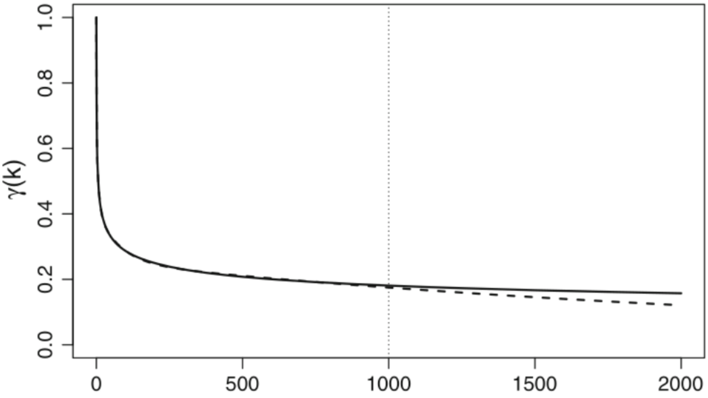

Continue...

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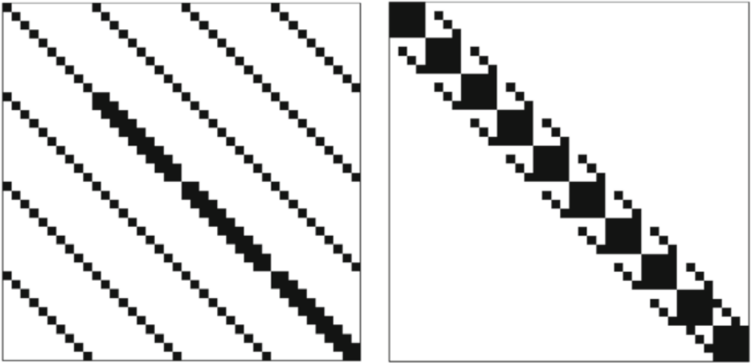

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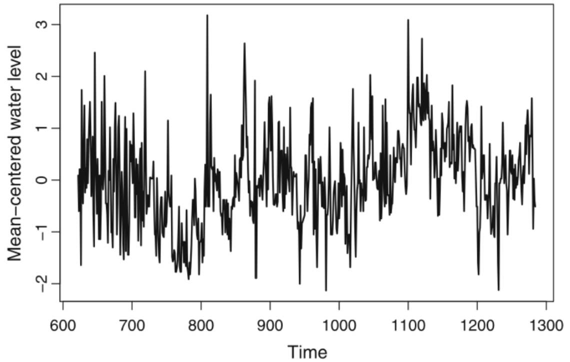
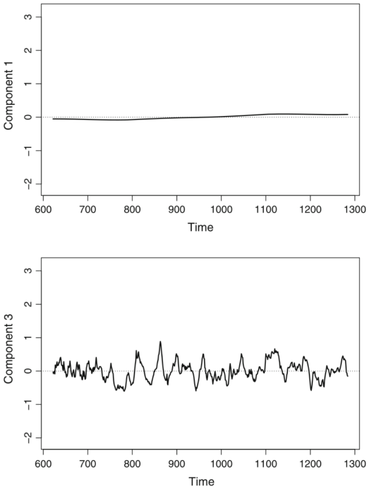


The Nile River data, particularly the yearly minima at Roda gauge (622–1284 AD) and annual streamflow (1871–1970), is a classic example of long-memory (or long-range dependence) in hydrology. Harold Edwin Hurst (1950s) identified that Nile fluctuations show high correlation over long periods, with a Hurst exponent $H \approx 0.9$, indicating persistent alternating periods of high and low flow, contrary to random, short-term models. AGU Publications +5

Continue...


Component 1
Component 2
Component 3
Component 4
Time
3
2
1
0
−1
−2
600
700
800
900
1000
1100
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FGN

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- Yes!
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- Precision isn't sold (nearly) at all!
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- If your model is *not* Markov, then be sure that you are doing it "right"
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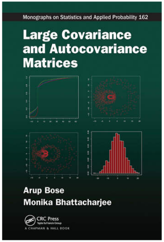

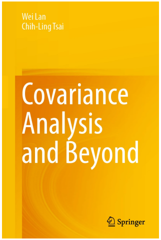

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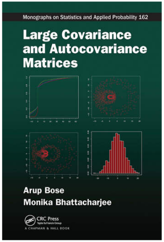

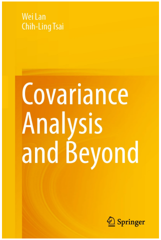

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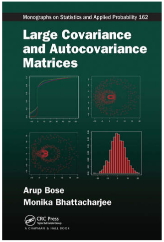

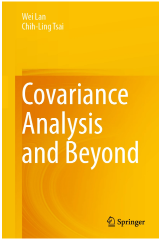

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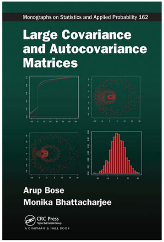

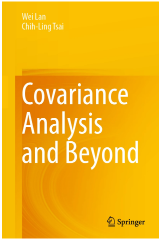

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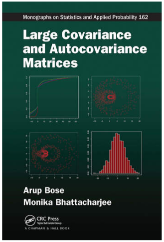

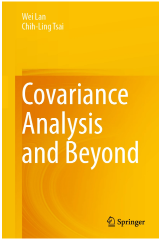

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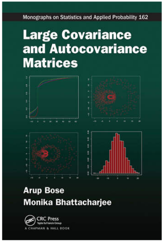

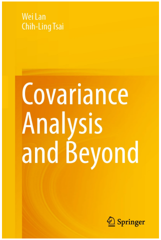

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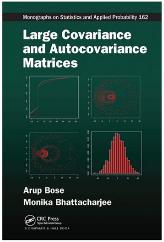

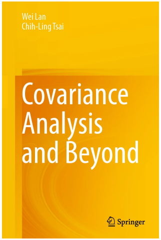

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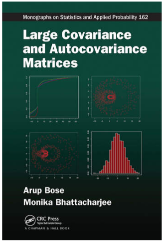

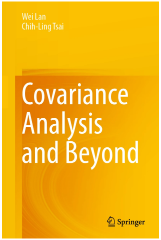

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- Variational Bayes
- Numerics for sparse (precision-)matrices
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What’s tomorrow?

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Thank you • شكرا

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